MJC-04

Unit-03 Differential Equation of the first order but not of the first degree

3.1 The most general form of a differential equation of the first order and of nth degree is

$$p^n + \varphi_1 \cdot p^{n-1} + \varphi_2 \cdot p^{n-2} + \cdots + \varphi_n = 0$$

where p stands for $\frac{dy}{dx}$ and $\varphi_1, \varphi_2, \varphi_3, \ldots, \varphi_n$ are functions of x and y.

In order to solve, The diff. eq. are of following types.

Type I (i) The left hand side of equation can be resolved into rational factors linear in the derivative; Solvable for p

Type II (ii) (a) Solvable for y
(b) solvable for x
(c) Clairaut's equations

Type I Left Hand sides resolvable into factors: solvable for p.

W.K. The nth degree and first order d. e.

$$p^n + \varphi_1 \cdot p^{n-1} + \varphi_2 \cdot p^{n-2} + \cdots + \varphi_n = 0 \quad \text{--- (1)}$$

Let Eq (1) can be resolved into factors

$$(p - A_1)(p - A_2)(p - A_3) \cdots (p - A_n) = 0$$

$$\Rightarrow p = A_1, \; p = A_2, \; p = A_3, \ldots, p = A_n$$

$$\Rightarrow f_1(x, y, c_1) = 0, \; f_2(x, y, c_2) = 0, \; f_3(x, y, c_3) = 0, \ldots f(x, y, c_n) = 0$$

$\therefore$ The complete solution is

$$f_1(x, y, c_1) \cdot f_2(x, y, c_2) \cdot f_3(x, y, c_3) \cdots f(x, y, c_n) = 0$$

Ex (1) Solve: $p^2 - p(e^x + e^{-x}) + 1 = 0$

Soln:- Given eqn:

$$p^2 - p(e^x + e^{-x}) + 1 = 0$$

$$\Rightarrow p^2 - pe^x - pe^{-x} + 1 = 0$$

$$\Rightarrow p(p - e^x) - e^{-x}\left(p - \frac{1}{e^{-x}}\right) = 0$$

$\Rightarrow (p-e^x)(p-e^{-x})=0$

$\Rightarrow p-e^x=0$ or $p=e^{-x}$

$\Rightarrow \cancel{p=e^x \text{ or } p=e^{-x}}$

$\Rightarrow \frac{dy}{dx}=e^x$ or $\frac{dy}{dx}=-e^{-x}$

$\Rightarrow dy=e^x.dx$ or $dy=-e^{-x}dx$

Taking Integration

$\int dy=\int e^x.dx$ or $\int dy=-\int e^{-x}dx$

$\Rightarrow y=e^x+c$ or $y=e^{-x}+c$

$\Rightarrow y-e^x-c=0$ or $y-e^{-x}-c=0$

The Complete solution

$(y-e^x-c)(y-e^{-x}-c)=0$

Ans

Ex(2) Solve: $p^2+2xp-3x^2=0$

Soln:- Given, $p^2+2xp-3x^2=0$

$\Rightarrow p^2+3xp-xp-3x^2=0$

$\Rightarrow p(p+3x)-x(p+3x)=0$

$\Rightarrow (p-x)(p+3x)=0$

$\therefore p=x$ or $p=-3x$

$\Rightarrow \frac{dy}{dx}=x$ or $\frac{dy}{dx}=-3x$

$\Rightarrow dy=xdx$ or $dy=-3xdx$

$\Rightarrow \int dy=\int xdx$ or $\int dy=-3\int xdx$

$\Rightarrow y=\frac{x^2}{2}+c$ or $y=-\frac{3x^2}{2}+c$

$\Rightarrow y-\frac{x^2}{2}-c=0$ or $y+\frac{3x^2}{2}-c=0$

The Complete Solution

$(y-\frac{x^2}{2}-c)(y+\frac{3x^2}{2}-c)=0.$

Ex(3) Solve: $p^2+p(2x-y)-2xy=0$

Soln:- Given, $p^2+2px-py-2xy=0$

$\Rightarrow p(p+2x)-y(p+2x)=0$

$\Rightarrow (p+2x)(p-y)=0$

$\therefore p=-2x$, or, $p=y$

$\Rightarrow \frac{dy}{dx}=-2x$ or $\frac{dy}{dx}=y$

$\Rightarrow dy=-2xdx$ or $\frac{dy}{y}=dx$

Taking Integration

$\Rightarrow \int dy=-2\int xdx$ or $\int\frac{dy}{y}=\int dx$

$\Rightarrow y=-\cancel{2}.\frac{x^2}{\cancel{2}}+c$ or $\log y=x+c$

$\Rightarrow y+x^2-c=0$ or $\log y-x-c=0$

The Complete solution

$(y+x^2-c)(\log y-x-c)=0.$

TYPE 2 Left side Not Resolvable into Factors.

~~① solvable for y~~

W.R. : ① Given, diff. eqn. $y = f(x, p)$ — ①

It is said to solvable for y

(ii) Diff. w.r. to x

$$\frac{dy}{dx} = \frac{d}{dx} f(x, p) = \phi\left(x, p, \frac{dp}{dx}\right)$$

$$\Rightarrow p = \phi\left(x, p, \frac{dp}{dx}\right) \Rightarrow F(x, y, c) = 0 \quad \text{—②}$$

It is solved by previous method in two ways

ⓐ Eliminate p

ⓑ solve ① & ②

Ex ① solve: $y = x\left\{\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2\right\}$

Soln:- Given,

$$y = x(p + p^2)$$

$$\Rightarrow y = px + p^2x \quad \text{—①}$$

Diff ① w.r. to x

$$\frac{dy}{dx} = x\frac{dp}{dx} + p + p^2 \cdot 1 + x \cdot 2p\frac{dp}{dx}$$

$$\Rightarrow \cancel{p} = x\frac{dp}{dx} + \cancel{p} + p^2 + 2px\frac{dp}{dx}$$

$$\Rightarrow x\frac{dp}{dx}(1+2p) = -p^2$$

$$\left(\frac{1+2p}{p^2}\right)dp = -\frac{dx}{x} \quad \{V.S.\}$$

Taking Integration

$$\int\left(\frac{1}{p^2} + \frac{2}{p}\right)dp = -\int\frac{dx}{x}$$

$$\Rightarrow \frac{p^{-1}}{-1} + 2\log p = -\log x + \log c$$

$$\Rightarrow -\frac{1}{p} + \log p^2 + \log x - \log c = 0$$

$$\Rightarrow \log\left(\frac{p^2x}{c}\right) = \frac{1}{p}$$

$$\Rightarrow \frac{p^2x}{c} = e^{1/p}$$

$$\Rightarrow x = \frac{ce^{1/p}}{p^2}$$

$$\Rightarrow x = cp^{-2}e^{1/p} \quad \text{—②}$$

The p-eliminating ① and ② gives the required solution.

x

dx

c

=c⇒

0.

∫dx

x+C

c⇒

⇒0.

Ex 2 Solve: $y = 2px + p^2$.

Soln:- Given, $y = 2px + p^2$ — (I)

Diff (I) w.r.to x

$$\frac{dy}{dx} = 2\left(p + x\frac{dp}{dx}\right) + 2p\cdot\frac{dp}{dx}$$

$$\Rightarrow p = 2p + 2x\frac{dp}{dx} + 2p\frac{dp}{dx}$$

$$\Rightarrow 2(x+p)\frac{dp}{dx} = -p$$

$$\Rightarrow \frac{2(x+p)}{-p} = \frac{dx}{dp}$$

$$\frac{dx}{dp} = -\frac{2}{p}x - 2$$

$$\Rightarrow \frac{dx}{dp} + \left(\frac{2}{p}\right)\cdot x = -2$$ which is linear,

$$I.F. = e^{\int P dy} = e^{\int \frac{2}{p} dy} = e^{2\log p} = p^2$$

The general Solution

$$x \times IF = \int Q \times IF\, dp$$

$$x \times p^2 = \int (-2)\cdot p^2 \cdot dp$$

$$x p^2 = -2\cdot\frac{p^3}{3} + c$$

$$\Rightarrow x = -\frac{2}{3}p + cp^{-2} \quad \text{— (II)}$$

From (I), $y = 2px + p^2$

$$\Rightarrow y = 2p\left(-\frac{2}{3}p + cp^{-2}\right) + p^2$$

$$\Rightarrow y = -\frac{4}{3}p^2 + 2cp^{-1} + p^2$$

$$\Rightarrow \boxed{y = 2cp^{-1} - \frac{1}{3}p^2}$$ This is the general Solution.

Ex 3 Solve: $y + px = x^4p^2$.

Soln:- Given,

$y = -px + x^4p^2$ — (I)

This is solvable for y

Diff (I) w.r.to x

$$\Rightarrow \frac{dy}{dx} = -\left(x\frac{dp}{dx} + p\frac{dx}{dx}\right) + \left(p^2\frac{dx^4}{dx} + x^4\cdot\frac{dp^2}{dx}\right)$$

$$\Rightarrow p = -x\frac{dp}{dx} - p + p^2\cdot 4x^3 + x^4\cdot 2p\cdot\frac{dp}{dx}$$

$$\Rightarrow \left(2p + x\frac{dp}{dx}\right) = 2px^3\left(x\frac{dp}{dx} + 2p\right)$$

$$\Rightarrow \left(2p + x\frac{dp}{dx}\right)\left(1 - 2px^3\right) = 0$$

$\therefore$ $2p + x\frac{dp}{dx} = 0$ or $1 - 2px^3 = 0$

$x\frac{dp}{dx} = -2p$ or $2px^3 = 1$

$\frac{dp}{p} = -2\frac{dx}{x}$ or $p = \frac{1}{2x^3}$

Taking Integration

$$\int\frac{dp}{p} = -2\int\frac{dx}{x}$$

$$\log p = -2\log x + c$$

$$\log p + \log x^2 = \log K$$

$$\log px^2 = \log K \Rightarrow px^2 = K \quad \text{— (2)}$$

Eliminating p between (1) & (2)

$$y + px = x^4 p^2$$

$$\Rightarrow y + \frac{K}{x^2}\cdot x = x^4\cdot\left(\frac{K}{x^2}\right)^2$$

$$\Rightarrow y + \frac{K}{x} = K^2$$

$$\Rightarrow xy + K = K^2x$$